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GENERAL PROJECTIVE THEORY OF SPACE CURVES*

BY

E. J. WILCZYNSKI†

The theory of the invariants of a linear differential equation is not new. LAGUERRE, BRIOSCHI, HALPHEN, FORSYTH, BOUTON, FANO and others have written important papers on the subject. But in almost all of these papers the point of view is essentially analytical. It is only in the brilliant contributions of HALPHEN that one finds the idea of founding a geometry of curves upon this theory. But even HALPHEN's papers do not give the ground work for a thorough comprehension of the subject. This is to be found in the geometric interpretation of the semi-covariants which we shall discuss in this paper. With this as a basis, the whole theory becomes clear and transparent, and HALPHEN's results can easily be connected with it.

I have myself shown in recent years, how the general projective theory of ruled surfaces depends upon the theory of the invariants and covariants of a system of two linear differential equations of the second order. The present paper is governed by similar ideas, and to some extent depends upon this other theory. It is the main purpose of this paper to bring up the general projective theory of curves, based upon the theory of invariants, to the same level of perfection as the corresponding theory of ruled surfaces. Some of the theorems which we shall find, have of course been known for a long time. But even for most of these our proofs will be new.

§ 1. *The invariants and covariants.*

We shall confine our attention to the linear differential equation of the fourth order

$$(1) \qquad y^{(4)} + 4p_1 y^{(3)} + 6p_2 y'' + 4p_3 y' + p_4 y = 0,$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \quad \text{etc.}$$

If we make the transformations

$$y = \lambda(x)\bar{y}, \qquad \xi = \xi(x),$$

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† Of the Carnegie Institution of Washington.

where λ and μ are arbitrary functions, we obtain another equation of the same form as (1). The functions of the coefficients of (1), which remain invariant under these transformations, are the *invariants*. If they involve also the functions y, y', y'', $y^{(3)}$, they are called *covariants*.

To determine these invariant functions, it is best according to LIE's theory to make use of infinitesimal transformations. But this has been done in a number of papers by other authors and need not therefore be repeated. We shall, however, need the finite transformations for the later parts of this paper, and shall, therefore, write them down.

Consider first the simpler transformation

$$y = \lambda(x)\bar{y},$$

of the dependent variable alone. We shall speak of the corresponding invariant functions as *seminvariants* and *semi-covariants*. Let

$$\bar{y}^{(4)} + 4\pi_1\bar{y}^{(3)} + 6\pi_2\bar{y}'' + 4\pi_3\bar{y}' + \pi_4\bar{y} = 0$$

be the transformed equation. Then

$$(2)\qquad \begin{aligned} \pi_1 &= \frac{\lambda' + p_1\lambda}{\lambda}, \\ \pi_2 &= \frac{\lambda'' + 2p_1\lambda' + p_2\lambda}{\lambda}, \\ \pi_3 &= \frac{\lambda^{(3)} + 3p_1\lambda'' + 3p_2\lambda' + p_3\lambda}{\lambda}, \\ \pi_4 &= \frac{\lambda^{(4)} + 4p_1\lambda^{(3)} + 6p_2\lambda'' + 4p_3\lambda' + p_4\lambda}{\lambda}, \end{aligned}$$

whence one may deduce the absolute seminvariants

$$(3)\qquad \begin{aligned} P_2 &= p_2 - p_1' - p_1^2, \\ P_3 &= p_3 - p_1'' - 3p_1p_2 + 2p_1^3, \\ P_4 &= p_4 - 4p_1p_3 - 3p_2^2 + 12p_1^2p_2 - 6p_1^4 - p_1^{(3)}, \end{aligned}$$

and the relative semi-covariants, besides y which is obviously itself a semi-covariant,

$$(4)\qquad \begin{aligned} z &= y' + p_1y, \\ \rho &= y'' + 2p_1y' + p_2y, \\ \sigma &= y^{(3)} + 3p_1y'' + 3p_2y' + p_3y. \end{aligned}$$

The absolute semi-covariants are z/y, ρ/y, σ/y. All other semi-covariants and seminvariants are functions of these and of the derivatives of P_2, P_3, P_4.

From (4) we deduce the following equations, which we shall use later:

$$
\begin{aligned}
y' &= -p_1 y + z, \\
z' &= -P_2 y - p_1 z + \rho, \\
\rho' &= -(P_3 - P_2')y - 2P_2 z - p_1\rho + \sigma, \\
\sigma' &= -(P_4 - P_3')y - 3(P_3 - P_2')z - 3P_2\rho - p_1\sigma,
\end{aligned} \tag{5}
$$

and also

$$
\begin{aligned}
y'' &= (2p_1^2 - p_2)y - 2p_1 z + \rho, \\
y^{(3)} &= (-p_3 + 6p_1p_2 - 6p_1^3)y + (-3p_2 + 6p_1^2)z - 3p_1\rho + \sigma, \\
y^{(4)} &= (-p_4 + 8p_1p_3 - 36p_1^2p_2 + 6p_2^2 + 24p_1^4)y \\
&\quad + (-4p_3 + 24p_1p_2 - 24p_1^3)z + (-6p_2 + 12p_1^2)\rho - 4p_1\sigma.
\end{aligned} \tag{6}
$$

We now proceed to make a transformation of the independent variable $\xi = \xi(x)$. We find, denoting the coefficients of the transformed equation by $\bar{p}_k$,

$$
\begin{aligned}
\bar{p}_1 &= \frac{1}{\xi'}(p_1 + \tfrac{3}{2}\eta), \\
\bar{p}_2 &= \frac{1}{(\xi')^2}[p_2 + 2\eta p_1 + \tfrac{1}{6}(4\mu + 9\eta^2)], \\
\bar{p}_3 &= \frac{1}{(\xi')^3}[p_3 + \tfrac{3}{2}\eta p_2 + (\mu + \tfrac{3}{2}\eta^2)p_1 + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)], \\
\bar{p}_4 &= \frac{1}{(\xi')^4}p_4,
\end{aligned} \tag{7}
$$

where we have put

$$
\eta = \frac{\xi''}{\xi'}, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2. \tag{8}
$$

We find further

$$
\begin{aligned}
\bar{z} &= \frac{1}{\xi'}(z + \tfrac{3}{2}\eta y), \\
\bar{\rho} &= \frac{1}{(\xi')^2}[\rho + 2\eta z + \tfrac{1}{6}(4\mu + 9\eta^2)y], \\
\bar{\sigma} &= \frac{1}{(\xi')^3}[\sigma + \tfrac{3}{2}\eta\rho + (\mu + \tfrac{3}{2}\eta^2)z + \tfrac{1}{4}(\mu' + 4\eta\mu + 3\eta^3)y],
\end{aligned} \tag{9}
$$

cogredient with (7).

Making use of these equations, we find

$$\bar{P}_2 = \frac{1}{(\xi')^2}[P_2 - \tfrac{5}{6}\mu],$$

$$(10)\quad \bar{P}_3 = \frac{1}{(\xi')^3}[P_3 - 3\eta P_2 - \tfrac{5}{4}\mu' + \tfrac{10}{4}\eta\mu],$$

$$\bar{P}_4 = \frac{1}{(\xi')^4}[P_4 - 6\eta P_3 - 4\eta' P_2 + 11\eta^2 P_2 + \tfrac{15}{2}\eta\mu' - \tfrac{15}{2}\eta^2\mu - \tfrac{3}{2}\mu'' + \tfrac{19}{6}\mu^2],$$

whence

$$\bar{P}_2' = \frac{1}{(\xi')^3}[P_2' - 2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu],$$

$$(11)\quad \bar{P}_2'' = \frac{1}{(\xi')^4}[P_2'' - 5\eta P_2' - 2\mu P_2 + 5\eta^2 P_2 - \tfrac{5}{6}\mu'' + \tfrac{5}{3}\mu^2 - \tfrac{25}{6}\mu\eta^2 + \tfrac{25}{6}\mu'\eta],$$

$$\bar{P}_3' = \frac{1}{(\xi')^4}[P_3' - 3\eta(P_2' + P_3) - 3\mu P_2 + \tfrac{15}{2}\eta^2 P_2 - \tfrac{5}{4}\mu'' + \tfrac{25}{4}\eta\mu' - \tfrac{25}{4}\mu\eta^2 + \tfrac{10}{4}\mu^2].$$

We find therefore the following invariants and covariants:

$$\theta_3 = P_3 - \tfrac{3}{2}P_2', \qquad \theta_4 = P_4 - 2P_3' + \tfrac{6}{5}P_2'' - \tfrac{6}{25}P_2^2,$$

$$\theta_{3\cdot 1} = 6\theta_3\theta_3'' - 7\theta_3'^2 - \tfrac{108}{5}P_2\theta_3^2,$$

$$(12)\qquad C_2 = 10z^2 - 15y\rho - 12P_2y^2,$$

$$C_3 = 10z^3 - 3C_2z - 9(5\sigma + 6P_2z + P_3y)y^2,$$

$$C_4 = 2\theta_3 z + \theta_3' y,$$

where the index indicates the weight. In denoting one invariant of weight 8 by $\theta_{3\cdot 1}$ we follow a notation due to FORSYTH. An invariant may be regarded as a covariant of degree zero. With this understanding, it suffices to say that the effect of the complete transformation

$$\bar{y} = \lambda(x)y, \qquad \xi = \xi(x),$$

upon a covariant of degree d and of weight w, is to transform it into $\bar{C}$, where

$$\bar{C} = \frac{\lambda^d}{(\xi')^w}C.$$

LIE's theory shows that all other invariants and covariants may be deduced from these by algebraic and differentiation processes.

§ 2. *Canonical forms.*

Equations (2) show that if we make the transformation

$$y = e^{-\int p_1 dx}\bar{y},$$

the coefficients of the resulting equation for $\bar{y}$ will be

$$\pi_1 = 0, \qquad \pi_2 = P_2, \qquad \pi_3 = P_3, \qquad \pi_4 = P_4 + 3P_2^2.$$

We shall say that the equation has been put into the *semi-canonical* form.

From (10) we see that if $\xi(x)$ be chosen so that

$$\eta' - \tfrac{1}{2}\eta^2 = \mu = \tfrac{6}{5}P_2, \tag{13}$$

in the resulting equation $\bar{P}_2$ will be zero. Since P_2 is a seminvariant, any transformation of the form $\bar{y} = \lambda y$ will not disturb the equation $P_2 = 0$, and we may again choose λ so as to make the coefficient of $d^3\bar{y}/d\xi^3$ vanish. It is therefore always possible to reduce the equation to the form

$$\frac{d^4\bar{y}}{d\xi^4} + 4\pi_3\frac{d\bar{y}}{d\xi} + \pi_4\bar{y} = 0,$$

which we shall call the LAGUERRE-FORSYTH *canonical form.* This is equivalent to assuming $p_1 = p_2 = 0$ in the original equation.

If $\theta_3 \neq 0$, we may transform the independent variable so as to make $\bar{\theta}_3 = 1$. In fact we have for an arbitrary transformation

$$\bar{\theta}_3 = \frac{1}{(\xi')^3}\theta_3.$$

If, therefore, we put

$$\xi = \int \sqrt[3]{\theta_3}\,dx, \tag{14}$$

$\bar{\theta}_3$ will be equal to unity. We may again by a transformation of the form $\bar{y} = \lambda y$ make p_1 vanish. The canonical form which is characterized by the conditions

$$p_1 = 0, \qquad \theta_3 = 1,$$

we may properly denote as the HALPHEN *canonical form.*

In our geometrical discussions only the quantity

$$\eta = \frac{\xi''}{\xi'},$$

not ξ itself will be of any importance. λ also is an unimportant factor which has no geometrical significance. Equation (13) shows, therefore, that the reduction to the LAGUERRE-FORSYTH form can always be accomplished in ∞^1 essentially different ways. It is important to remark that (13) is an equation of the RICCATI form, so that the cross-ratio of any four solutions is constant.

The HALPHEN form on the other hand can be obtained in just one way, if it exists at all, i. e., if $\theta_3 \neq 0$. If θ_3 vanishes, θ_4 may be reduced to unity unless it also is equal to zero. The case when both θ_3 and θ_4 vanish, is especially simple. The LAGUERRE-FORSYTH form reduces to

$$\frac{d^4\bar{y}}{d\xi^4} = 0.$$

If two equations of the form (1) can be transformed into each other by a transformation of the kind here considered, we shall call them equivalent. Clearly, for equivalent equations, the corresponding absolute invariants are equal.

If equation (1) is given, the invariants θ_3, θ_4, $\theta_{3.1}$, etc., are known functions of x. Conversely, equations (12) show that if θ_3, θ_4, $\theta_{3.1}$ are given as arbitrary functions of x, provided that $\theta_3 \neq 0$, P_2, P_3 and P_4 are determined uniquely. If $\theta_3 = 0$, then $\theta_{3.1} = 0$ also, and we must assign a further condition. The function

$$\theta_{4.1} = 8\theta_4\theta_4'' - 9(\theta_4')^2 - \frac{80}{3} P_2\theta_4^2 \tag{15}$$

is also an invariant. If $\theta_3 = 0$, and θ_4, $\theta_{4.1}$ are given, P_2, P_3, P_4 are determined uniquely. If both θ_3 and θ_4 vanish, all invariants are zero, and the equation may be reduced to the form

$$\frac{d^4\eta}{d\xi^4} = 0.$$

As we may always assume that $p_1 = 0$, we see that *the differential equation* (1) *is essentially determined when its invariants are given as functions of* x.

The LAGRANGE adjoint of (1) is

$$\begin{aligned} u^{(4)} - 4p_1u^{(3)} + 6(p_2 - 2p_1')u'' - 4(p_3 - 3p_2' + 3p_1'')u' \\ + (p_4 - 4p_3' + 6p_2'' - 4p_1^{(3)})u = 0. \end{aligned} \tag{16}$$

If $y_1, \cdots, y_4$ constitute a fundamental system of (1), the minors of $x_1, \cdots, x_4$ in the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \end{vmatrix}$$

multiplied by a common factor, which does not interest us, form a fundamental system of (16).

If we denote the seminvariants of (16) by Π_2, Π_3, Π_4, we have

$$(17) \qquad \Pi_2 = P_2, \qquad \Pi_3 = -P_3 + 3P_2', \qquad \Pi_4 = P_4 - 4P_3' + 6P_2'',$$

whence follows reciprocally

$$P_2 = \Pi_2, \qquad P_3 = -\Pi_3 + 3\Pi_2', \qquad P_4 = \Pi_4 - 4\Pi_3' + 6\Pi_2''.$$

The invariants of (16) differ from the invariants of (1) only in this that the sign of θ_3 is changed.

§ 3. *Geometrical interpretation.*

If the functions $y_1, \cdots, y_4$ constitute a fundamental system of (1) we may interpret them as the homogeneous coördinates of a point P_y of a curve C_y in ordinary space. The coefficients p_1, p_2, p_3, p_4 of (1) are invariants of the general projective group. The transformation $y = \lambda \bar{y}$ does not change the ratios $y_1 : y_2 : y_3 : y_4$, and therefore leaves the curve C_y invariant. The transformation $\xi = \xi(x)$ merely changes the parameter in terms of which the coördinates are expressed. It is clear therefore that any system of equations invariant under these transformations expresses a projective property of the curve C_y in the vicinity of the point P.

The LAGRANGE adjoint of (1) may be taken to represent the same curve in tangential coördinates, or else a reciprocal curve in point coördinates.

We may therefore state the results of § 2 as follows: *If the invariants of a curve are given as functions of x, the curve is determined except for projective transformations. If the invariants of two curves, except those of weight three, are respectively equal to each other, while the invariants of weight three differ only in sign, the two curves are dualistic to each other. Those curves are self-dual for which $\theta_3 = 0$.*

Moreover these latter curves are the only curves which are self-dual in the restricted sense that a dualistic transformation exists which converts every point of the curve into the tangent plane of that point, and vice versâ, while every tangent is converted into itself.

If we put $y = y_k (k = 1, 2, 3, 4)$ into the expressions for z, ρ, σ we obtain three other points P_z, P_ρ, P_σ, which describe curves C_z, C_ρ, C_σ as x varies, curves which are closely connected with C_y. P_z is clearly a point on the tangent of C_y constructed at P_y; P_ρ is in the plane osculating C_y at P_y, while P_σ is outside of this plane. These four points are never coplanar except at those exceptional points of C_y whose osculating planes are stationary, i. e., have more than three consecutive points in common with the curve.

In order to study the curve C_y in the vicinity of P_y, it will therefore be convenient to introduce the tetrahedron $P_y P_z P_\rho P_\sigma$ as tetrahedron of reference, with the further convention that if any expressions of the form

$$u_k = \alpha_1 y_k + \alpha_2 z_k + \alpha_3 \rho_k + \alpha_4 \sigma_k \qquad (k = 1, 2, 3, 4)$$

offer themselves the coördinates of the corresponding point P_u shall be

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

In writing u_k the index k may be suppressed, so that a single expression

$$\alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma$$

represents the point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in this special system of coördinates.

As the independent variable x is changed, the tetrahedron of reference is changed in accordance with equations (9). P_y of course remains the same; P_z is changed into $P_{\bar{z}}$, which may obviously be any point on the tangent: etc. Thus while an arbitrary transformation of the parameter x does not affect the curve C_y itself, it does very materially affect the semi-covariant curves C_z, C_ρ and C_σ. It is clear however that two transformations $\xi = \xi(x)$, for which $\eta = \xi''/\xi'$ has the same value, are geometrically equivalent. We may also, without affecting the position of the points P_z, P_ρ, P_σ, assume that (1) is written in the semi-canonical form, so that $p_1 = 0$. For, in order to put (1) into the semi-canonical form, we need only multiply y by a certain factors λ, which will then also appear multiplied into the semi-covariants z, ρ and σ.

Let us then assume $p_1 = 0$. We shall have $z = y'$. If we differentiate (1) and eliminate y between the resulting equation and (1), we shall find

$$
\begin{aligned}
&(P_4 + 3P_2^2)z^{(4)} - (P_4' + 6P_2P_2')z^{(3)} + 6P_2(P_4 + 3P_2^2)z'' \\
(18)\qquad &+ [(6P_2' + 4P_3)(P_4 + 3P_2^2) - 6P_2(P_4' + 6P_2P_2')]z' \\
&+ [(4P_3' + P_4 + 3P_2^2)(P_4 + 3P_2^2) - 4P_3(P_4' + 6P_2P_2')]z = 0,
\end{aligned}
$$

if $P_4 + 3P_2^2 \neq 0$. If $P_4 + 3P_2^2 = 0$ we find

$$(19)\qquad z^{(3)} + 6P_2z' + 4P_3z = 0.$$

Equation (18) determines the curve C_z in the same way as (1) determines C_y. But if $P_4 + 3P_2^2 = 0$, z satisfies (19) showing that the curve C_z is in this case a plane curve. Therefore, if the variable ξ be so chosen as to make $P_4 + 3P_2^2 = 0$, the corresponding curve $C_{\bar{z}}$ is a plane section of the developable surface whose cuspidal edge is C_y. In harmony with this, equations (10) show that the most general value of η, which satisfies the condition $P_4 + 3P_2^2 = 0$, contains three arbitrary constants, as it should since there are ∞^3 planes in space.

We shall need to consider the ruled surfaces generated by those edges of our tetrahedron which meet in P_y. Of these we know one immediately, namely the developable which has C_y as its edge of regression, and of which P_yP_z is a generator. The ruled surface generated by P_yP_ρ clearly has C_y as an asymptotic curve; for, the plane $P_yP_\rho P_z$ is both osculating plane of C_y at P_y, and

tangent plane of the surface at P_y. If we assume $p_1 = 0$, this ruled surface may be studied by means of the equations

$$(20)\qquad \begin{aligned} y'' + P_2 y - \rho &= 0, \\ \rho'' + (4P_3 - 2P_2')y' + (P_4 - P_2'' - 2P_2^2)y + 5P_2\rho &= 0, \end{aligned}$$

in accordance with the general theory of ruled surfaces as developed in former papers of the author. To prove (20) we need only differentiate twice the expression for ρ, express $y^{(3)}$ and $y^{(4)}$ in terms of y, z, ρ, σ, and eliminate z and σ.

The ruled surface generated by $P_y P_\sigma$ is especially important. We have,

$$(21)\qquad \sigma = y^{(3)} + 3p_1 y^{(2)} + 3p_2 y' + p_3 y,$$

whence

$$\sigma' = y^{(4)} + 3p_1 y^{(3)} + (3p_1' + 3p_2)y'' + (3p_2' + p_3)y' + p_3' y,$$

$$(22)\qquad \begin{aligned} \sigma'' = y^{(5)} &+ 3p_1 y^{(4)} + 3(2p_1' + p_2)y^{(3)} \\ &+ (3p_1'' + 6p_2' + p_3)y'' + (3p_2'' + 2p_3')y' + p_3'' y. \end{aligned}$$

From (21) we find

$$y^{(3)} = \sigma - 3p_1 y'' - 3p_2 y' - p_3 y,$$

$$(23)\qquad \begin{aligned} y^{(4)} = \sigma' &- 3p_1\sigma - 3(p_1' + p_2 - 3p_1^2)y'' \\ &- (3p_2' + p_3 - 9p_1p_2)y' - (p_3' - 3p_1p_3)y. \end{aligned}$$

If we substitute these values in (1) we obtain the equation

$$(24)\qquad \begin{aligned} 3(p_2 - p_1' - p_1^2)y'' &+ 3(p_3 - p_2' - p_1p_2)y' + \sigma' \\ &+ (p_4 - p_1p_3 - p_3')y + p_1\sigma = 0, \end{aligned}$$

where the coefficient of y'' is $3P_2$.

Let us differentiate both members of this equation, and eliminate y'' and $y^{(3)}$ by means of (23) and (24). We shall find

$$(25)\qquad \begin{aligned} 3P_2\sigma'' = (q_3 r_3 &+ 3P_2 q_4)y' + (3P_2 q_1 - q_3)\sigma' \\ &+ (r_4 q_3 + 3P_2 q_5)y + (3P_2 q_2 - p_1 q_3)\sigma, \end{aligned}$$

where

$$(26)\qquad \begin{aligned} &q_1 = -p_1, \qquad q_2 = 2p_1' - 3p_2 + 3p_1^2, \qquad q_3 = -3P_3 + 3p_1P_2, \\ &q_4 = -2p_3' + 3p_2'' + 3p_1p_2' - 6p_1'p_2 - p_4 + p_1p_3 + 9p_2^2 - 9p_1^2p_2, \\ &q_5 = -(p_4' - p_3'' - p_1p_3' - p_1'p_3) + 3p_3(p_2 - p_1' - p_1^2), \\ &r_3 = -3(p_3 - p_2' - p_1p_2), \\ &r_4 = -(p_4 - p_3' - p_1p_3). \end{aligned}$$

Equations (24) and (25) define the ruled surface generated by $P_y P_\sigma$. If we assume $p_1 = 0$, we find

$$
(27)\qquad
\begin{aligned}
&y'' + p_{11}y' + p_{12}\sigma' + q_{11}y + q_{12}\sigma = 0,\\
&\sigma'' + p_{21}y' + p_{22}\sigma' + q_{21}y + q_{22}\sigma = 0,
\end{aligned}
$$

where

$$
p_{11} = \frac{P_3 - P_2'}{P_2}, \qquad p_{12} = \frac{1}{3P_2}, \qquad q_{11} = \frac{P_4 + 3P_2^2 - P_3'}{3P_2}, \qquad q_{12} = 0,
$$

$$
(28)\qquad
\begin{aligned}
&p_{21} = \frac{1}{P_2}[-3P_3^2 + 3P_3P_2' + 2P_2P_3' - 3P_2P_2'' + P_2P_4 - 6P_2^3],\\
&p_{22} = -\frac{P_3}{P_2},\\
&q_{21} = \frac{1}{P_2}[-P_3P_4 - 6P_2^2P_3 + P_3P_3' + P_2P_4' + 6P_2^2P_2' - P_2P_3''],\\
&q_{22} = 3P_2.
\end{aligned}
$$

If (1) is written in the LAGUERRE-FORSYTH form, $P_2 = 0$. In that case the two equations (27) reduce to the single equation

$$
(29)\qquad P_3y' + \tfrac{1}{3}\sigma' + \tfrac{1}{3}(P_4 - P_3')y = 0,
$$

which proves that *in this case the surface generated by $P_y P_\sigma$ is developable.* For in this case the tangents constructed respectively to C_y at P_y and to C_σ at P_σ are coplanar. Moreover, only if $P_2 = 0$ will the surface generated by $P_y P_\sigma$ be a developable.

Let

$$
\tau = \lambda y + \mu\sigma
$$

represent the point at which $P_y P_\sigma$ intersects the edge of regression of the developable. Then, since $P_y P_\sigma$ must be tangent to the edge of regression, we shall have $\tau' = \alpha y + \beta\sigma$, or

$$
(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' + \mu\sigma' = 0.
$$

But according to (29)

$$
\sigma' = -(P_4 - P_3')y - 3P_3y',
$$

so that

$$
(\lambda' - \alpha)y + (\mu' - \beta)\sigma + \lambda y' - \mu[(P_4 - P_3')y + 3P_3y'] = 0,
$$

where for y' we could also write z. Such a relation between P_y, P_z, P_σ would, however, make these three points collinear, and, therefore, P_y, P_z, P_ρ, P_σ coplanar, unless all of the coefficients are zero. We have seen, however, that this

can happen only for such points P_y at which the osculating plane is stationary. We must have, therefore,

$$\lambda' - \alpha - \mu(P_4 - P_3') = 0, \qquad \mu' - \beta = 0, \qquad \lambda - 3\mu P_3 = 0,$$

whence

$$\lambda = 3\mu P_3, \qquad \beta = \mu', \qquad \alpha = \lambda' - \mu(P_4 - P_3').$$

We see, therefore, that

$$\tau = 3P_3 y + \sigma \tag{30}$$

represents the edge of regression of the developable to which the ruled surface generated by $P_y P_\sigma$ reduces when $P_2 = 0$.

If $p_1 = 0$ and $P_2 = 0$, equations (2) and (10) show that the most general transformations of the variables, which do not disturb these conditions, satisfy the equations

$$\frac{\lambda'}{\lambda} + \frac{3}{2}\frac{\xi''}{\xi'} = 0, \qquad \mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

which give on integration

$$\lambda = \frac{C}{(\xi')^{\frac{3}{2}}}, \qquad \eta = \frac{-2c}{1+cx}.$$

If we transform τ under this assumption, we find that it is converted into

$$\bar{\tau} = \frac{1}{(\xi')^3\lambda}\left[\sigma + \tfrac{3}{2}\eta\rho + \tfrac{3}{2}\eta^2 z + (\tfrac{3}{4}\eta^3 + 3P_3)y\right], \tag{30a}$$

where η may have any numerical value.

Let us recapitulate. *The ruled surfaces generated by $P_y P_{\bar{\sigma}}$ are infinite in number. Their general expression involves an arbitrary function η. Among these surfaces there exists a single infinity of developables. If $P_2 = 0$, the surface generated by $P_y P_\sigma$ is one of these, and the locus of P_τ is its edge of regression, where*

$$\tau = 3P_3 y + \sigma, \tag{30}$$

P_τ being the point where $P_y P_\sigma$ intersects the edge of regression. If we construct all the ∞^1 lines $P_y P_{\bar{\sigma}}$ through P_y, which are generators of the above mentioned family of developables, and mark upon each of them the point $P_{\bar{\tau}}$ where it intersects the cuspidal edge of the developable to which it belongs, the locus of these points is a twisted cubic curve. The equations of this curve referred to a parameter η and to the fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ are

$$x_1 = 3P_3 + \tfrac{3}{4}\eta^3, \qquad x_2 = \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1. \tag{31}$$

We shall see later that this cubic has five consecutive points in common with the curve C_y at P_y, i. e., that it has at this point with C_y a contact of the fourth

order. We shall speak of it as the *torsal cubic of* P_y, on account of its connection with the developables which we have just been considering.

Equations (31) give the parametric equations of the torsal cubic referred to a special tetrahedron of reference for which $P_2 = 0$. We shall need its equations in a more general form. These may be easily obtained. Consider the expression

$$(32) \qquad \lambda = (3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2 t + \tfrac{3}{4}t^3)y + (\tfrac{6}{5}P_2 + \tfrac{3}{2}t^2)z + \tfrac{3}{2}t\rho + \sigma$$

in which t may for the moment be regarded as a parameter independent of x. Denote by $\bar{\lambda}$ the corresponding expression formed from the quantities $\bar{P}_2$, $\bar{P}_3$, etc., $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ after the general transformation $\xi = \xi(x)$. We shall find that $(\xi')^3\bar{\lambda}$ is equal to λ after t has been replaced by $t\xi' + \eta = t_1$. But of course this transformation may be chosen so as to make $\bar{P}_2 = 0$, which would make $\bar{\lambda}$ identical with $\bar{\tau}$ except for the notation.

We see, therefore, that the expression λ, *or the equations*

$$(33) \qquad \begin{aligned} x_1 &= 3\theta_3 + \tfrac{3}{10}P_2' + \tfrac{6}{5}P_2\eta + \tfrac{3}{4}\eta^3, \\ x_2 &= \tfrac{6}{5}P_2 + \tfrac{3}{2}\eta^2, \qquad x_3 = \tfrac{3}{2}\eta, \qquad x_4 = 1, \end{aligned}$$

represent the torsal cubic referred to the fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ *when this is chosen in as general a way as is compatible with its definition.*

If in (32) t is chosen as a function of x, as x varies we obtain a curve on the surface formed by the totality of torsal cubics. If in particular t satisfies as function of x the differential equation

$$t' - \tfrac{1}{2}t^2 = \tfrac{6}{5}P_2,$$

we obtain the cuspidal edge of one of the developables.

§ 4. *The osculating cubic, conic and linear complex.*

A space cubic is determined by six of its points provided that no four of these points are coplanar. If, therefore, we take upon C_y, besides P_y, five other points, we shall in general obtain a perfectly definite space cubic determined by these six points. As these points approach coincidence with P_y, this cubic will in general approach a limit, which shall be called the *osculating cubic*. We proceed to find its equations.

Let P_y correspond to the value of $x = a$, which we shall suppose is an ordinary point for our differential equation. Then y may be developed by Taylor's theorem into a series proceeding according to powers of $x - a$. By putting $x - a = x'$ the development will be in powers of x'. We may therefore assume in the first place that $a = 0$. Let us assume, further, that $p_1 = 0$ and $P_2 = 0$. Then we shall have from (5) and (6),

$$
(34)\quad \begin{aligned} & y' = z, \qquad y'' = \rho, \qquad y^{(3)} = -P_3 y + \sigma, \qquad y^{(4)} = -P_4 y - 4P_3 z, \\ & y^{(5)} = -P_4' y - (P_4 + 4P_3')z - 4P_3 \rho. \end{aligned}
$$

In accordance with the definition of our coördinates, we denote the coefficients of y, z, ρ, σ in this expansion carried as far as x^5, by y_1, y_2, y_3, y_4. We may of course multiply these quantities by a common factor since the coördinates are homogeneous. We shall multiply by 120 so as to clear of fractions. This gives

$$
(35)\quad \begin{aligned} y_1 &= 120 - 20P_3 x^3 - 5P_4 x^4 - P_4' x^5 + \cdots, \\ y_2 &= 120x - 20P_3 x^4 - (4P_3' + P_4)x^5 + \cdots, \\ y_3 &= 60x^2 - 4P_3 x^5 + \cdots, \\ y_4 &= 20x^3 + \cdots. \end{aligned}
$$

We see at once that the following equations are exact up to terms no higher than the fifth order:

$$
(36)\qquad 3y_2 y_4 - 2y_3^2 = 0, \qquad 5(2y_1 y_3 - y_2^2) - 6P_3 y_3 y_4 = 0.
$$

These equations must be satisfied by the coördinates of any point of the osculating cubic, since this must have contact of the fifth order with C_y at P_y. *They are therefore its equations, referred to this special tetrahedron of reference. In terms of a parameter t we may write*

$$
(37)\quad x_1 = 15 + 12P_3 t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3.
$$

The equation

$$
3x_2 x_4 - 2x_3^2 = 0
$$

is that of a cone whose vertex is P_y and which contains the osculating cubic. It may also be obtained by determining that cone of the second order with its vertex at P_y which has the closest possible contact with C_y, viz. contact of the fifth order. We shall speak of it as the *osculating cone*. We notice at once that the torsal cubic also lies upon the osculating cone. This is shown by equations (31) which are referred to the same system of coördinates as that employed here. If we put $\eta = 1/t$ in (31) and if we multiply by $20t^3$, (31) becomes

$$
(31a)\quad x_1 = 15 + 60P_3 t^3, \qquad x_2 = 30t, \qquad x_3 = 30t^2, \qquad x_4 = 20t^3,
$$

which differs from (37) only in having $5P_3$ in place of P_3.

By a method of reasoning precisely similar to that of the last paragraph, we find that *the expression*

$$
(38)\quad (12P_3 - 12P_2' + 24P_2\tau + 15\tau^3)y + 20(\tfrac{6}{5}P_2 + \tfrac{3}{2}\tau^2)z + 30\tau\rho + 20\sigma
$$

represents an arbitrary point of the osculating cubic when the tetrahedron of reference is not restricted to the condition $P_2 = 0$. For this expression remains invariant under the general transformation $\xi = \xi(x)$, and reduces to (37) for $P_2 = 0$, if $\tau = 1/t$.

The equation of the plane, which osculates the osculating cubic at the point whose parameter is τ, turns out to be

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

where

$$(39) \qquad \begin{gathered} u_1 = -20, \qquad u_2 = 30\tau, \qquad u_3 = 16P_2 - 30\tau^2, \\ u_4 = 12P_3 - 12P_2' - 36P_2\tau + 15\tau^3. \end{gathered}$$

For every value of τ this intersects the osculating plane, $x_4 = 0$, in a straight line

$$-20x_1 + 30\tau x_2 + (16P_2 - 30\tau^2)x_3 = 0.$$

The envelope of these lines will be obtained by eliminating τ between this equation and that obtained from it by partial differentiation with respect to τ; the latter equation is

$$30x_2 - 60\tau x_3 = 0.$$

We thus find

$$(40) \qquad -40x_1x_3 + 15x_2^2 + 32P_2x_3^2 = 0,$$

the equation of the osculating conic, which may be defined as a part of the intersection of the developable of the osculating cubic with the osculating plane. The other part of this intersection is the tangent, which must be counted twice.

It is not without interest to verify that (37) represents the osculating cubic by another method. We have from (35) as non-homogeneous coördinates of the points of C_y in the vicinity of P_y

$$(41) \qquad \begin{gathered} \frac{y_2}{y_1} = x - \tfrac{1}{120}(P_3' - 4P_4)x^5 + \cdots, \qquad \frac{y_3}{y_1} = \tfrac{1}{2}x^2 + \tfrac{1}{20}P_3x^5 + \cdots, \\ \frac{y_4}{y_1} = \tfrac{1}{6}x^3 + \cdots. \end{gathered}$$

From (37) we find for the points of the osculating cubic

$$\frac{x_2}{x_1} = 2t - \tfrac{8}{5}P_3t^4 + \cdots, \qquad \frac{x_3}{x_1} = 2t^2 - \tfrac{8}{5}P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

If we put

$$t = \tfrac{1}{2}x[1 + \tfrac{1}{10}P_3x^3 - \tfrac{1}{30}(P_3' + P_4)x^4 + \cdots],$$

these two expansions coincide up to terms of the fifth order. For the torsal cubic we have according to $(31a)$,

$$\frac{x_2}{x_1} = 2t - 8P_3t^4 \cdots, \qquad \frac{x_3}{x_1} = 2t^2 - 8P_3t^5 + \cdots, \qquad \frac{x_4}{x_1} = \tfrac{4}{3}t^3 + \cdots.$$

If we put into these equations

$$t = \tfrac{1}{2}x(1 + ax + bx^2 + cx^3 + dx^4 + \cdots),$$

we find that these expansions will agree with (41) up to terms of the fourth order for

$$a = 0, \qquad b = 0, \qquad c = \tfrac{1}{2}P_3.$$

but that it is impossible to make them agree with (41) any farther unless $P_3 = 0$. *In general, therefore, the torsal cubic has with C_y a contact of the fourth order. Only if $\theta_3 = 0$ may the order of contact be higher. In that case the torsal and osculating cubics coincide.*

We proceed to deduce the equation of the *osculating linear complex*, i. e., of that linear complex determined by five consecutive tangents of the curve. We assume again $p_1 = 0$ and $P_2 = 0$. Denote by Y and Z the expansions of y and z in the vicinity of P_y. Then we have up to terms of the fourth order

$$Y = y(1 - \tfrac{1}{6}P_3x^3 - \tfrac{1}{24}P_4x^4) + z(x - \tfrac{1}{6}P_3x^4) + \tfrac{1}{2}\rho x^2 + \tfrac{1}{6}\rho x^3,$$

$$Z = y(-\tfrac{1}{2}P_3x^2 - \tfrac{1}{6}P_4x^3 - \tfrac{1}{24}P_4'x^4) + z[1 - \tfrac{2}{3}P_3x^3 - \tfrac{1}{24}(4P_3' + P_4)x^4]$$
$$+ \rho(x - \tfrac{1}{6}P_3x^4) + \sigma\tfrac{1}{2}x^2.$$

If we denote the coefficients of y, z, ρ, σ in these two expressions by $y_1, \cdots, y_4$ and $z_1, \cdots, z_4$, respectively, the Plückerian line-cöordinates of the tangent will be

$$\omega_{ik} = y_iz_k - y_kz_i,$$

whence

$$\omega_{12} = 1 - \tfrac{1}{6}P_3x^3 + \tfrac{1}{8}P_4x^4 + \cdots, \quad \omega_{13} = x - \tfrac{1}{12}P_3x^4 + \cdots, \quad \omega_{14} = \tfrac{1}{2}x^2 + \cdots,$$

$$\omega_{23} = \tfrac{1}{2}x^2 + \cdots, \qquad \omega_{24} = \tfrac{1}{3}x^3 + \cdots, \qquad \omega_{34} = \tfrac{1}{12}x^4 + \cdots.$$

Therefore, *the equation of the osculating linear complex, referred to the special tetrahedron of reference, is*

$$\omega_{14} - \omega_{23} = 0. \tag{42}$$

We might have obtained this complex in another way. For, it is clear that the null-system of the osculating cubic will be the same as that determined by the osculating linear complex. We shall, instead, set up the null-system of the torsal cubic in its general form. We shall see that *the linear complex determined by the torsal cubic coincides with the osculating linear complex.*

We have the equations of the torsal cubic

$$\begin{aligned} x_1 &= 60\theta_3 + 6P_2' + 24P_2\eta + 15\eta^3, \\ x_2 &= 24P_2 + 30\eta^2, \qquad x_3 = 30\eta, \qquad x_4 = 20. \end{aligned} \tag{43}$$

The coördinates of the plane, which osculates the torsal cubic at the point whose parameter is η, are

$$(44)\qquad \begin{aligned} &u_1 = -180, \qquad u_2 = 270\eta, \qquad u_3 = 144P_2 - 270\eta^2, \\ &u_4 = 540\theta_3 + 54P_2' - 324P_2\eta + 135\eta^3. \end{aligned}$$

If we put in (43) $\eta = \eta_k\,(k = 1, 2, 3)$ we obtain three points on the cubic. The coördinates of their plane must be proportional to

$$(45)\qquad \begin{aligned} &v_1 = -180, \qquad v_2 = 90(\eta_1 + \eta_2 + \eta_3), \\ &v_3 = 144P_2 - 90(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ &v_4 = 540\theta_3 + 54P_2' - 108P_2(\eta_1 + \eta_2 + \eta_3) + 135\eta_1\eta_2\eta_3, \end{aligned}$$

for each of these expressions must be a symmetric function of η_1, η_2, η_3 of order not higher than the third, and for $\eta_1 = \eta_2 = \eta_3 \doteq \eta$ we must have v_k proportional to u_k. Similarly, the point in which the three osculating planes at η_1, η_2, η_3 intersect, must have its coördinates proportional to

$$(46)\qquad \begin{aligned} &\omega x_1 = 60\theta_3 + 6P_2' + 8P_2(\eta_1 + \eta_2 + \eta_3) + 15\eta_1\eta_2\eta_3, \\ &\omega x_2 = 24P_2 + 10(\eta_2\eta_3 + \eta_3\eta_1 + \eta_1\eta_2), \\ &\omega x_3 = 10(\eta_1 + \eta_2 + \eta_3), \qquad \omega x_4 = 20. \end{aligned}$$

If we eliminate η_1, η_2, η_3 between (45) and (46) and change slightly the factor of proportionality, we find

$$(47a)\qquad \begin{aligned} &\omega v_1 = -x_4, \qquad \omega v_3 = -x_2 + 2P_2x_4, \\ &\omega v_2 = +x_3, \qquad \omega v_4 = +x_1 - 2P_2x_3, \end{aligned}$$

or

$$(47b)\qquad \begin{aligned} &\omega' x_1 = +2P_2v_2 + v_4, \qquad \omega' x_3 = +v_2, \\ &\omega' x_2 = -2P_2v_1 - v_3, \qquad \omega' x_4 = -v_1, \end{aligned}$$

as the equations of the null-system defined by the torsal cubic.

A point y_1, y_2, y_3, y_4 lies in the plane corresponding to x_1, x_2, x_3, x_4 if

$$\sum_{k=1}^{4} v_k y_k = 0.$$

Therefore, the lines which pass through the point x_1, x_2, x_3, x_4 and lie in the plane corresponding to it in the null-system, satisfy the equation

$$(48)\qquad \omega_{14} - 2P_2\omega_{34} - \omega_{23} = 0.$$

If the tetrahedron of reference be so chosen as to make $P_2 = 0$, this equation is identical with (42). *Therefore* (48) *represents the osculating linear complex*

when the tetrahedron of reference is general. The osculating and torsal cubics are curves of this complex.

If P_2 is finite the complex (48) is not special. We see therefore that only those values of x for which $P_2 = \infty$ can give points of the curve at which five consecutive tangents have a straight line intersector.

Let us proceed to deduce the equation of the osculating linear complex belonging to a point of C_y infinitesimally close to P_y. If we change x by an infinitesimal amount δx, we find for the coördinates of the vertices of the new tetrahedron of reference

$$\begin{aligned}
\bar{y} &= y + y'\delta x = (1 - p_1 \delta x) y + z\delta x, \\
\bar{z} &= z + z'\delta x = - P_2 \delta x \cdot y + (1 - p_1 \delta x) z + \rho\delta x, \\
\bar{\rho} &= \rho + \rho'\delta x = - (P_3 - P_2')\delta x \cdot y - 2P_2 \delta x \cdot z + (1 - p_1 \delta x)\rho + \sigma\delta x, \\
\bar{\sigma} &= \sigma + \sigma'\delta x = - (P_4 - P_3')\delta x \cdot y - 3(P_3 - P_2')\delta x \cdot z \\
&\qquad - 3P_2 \delta x \cdot \rho + (1 - p_1 \delta x)\sigma.
\end{aligned}$$

Therefore, if a point has the coördinates $\bar{x}_1, \cdots, \bar{x}_4$ in the new system of coördinates, and $x_1, \cdots, x_4$ in the old, we shall have

$$x_1 = (1 - p_1 \delta x)\bar{x}_1 - P_2 \delta x \cdot \bar{x}_2 - (P_3 - P_2')\delta x \cdot \bar{x}_3 - (P_4 - P_3')\delta x \cdot \bar{x}_4,$$

. . . .

Therefore the infinitesimal changes in the coördinates in the sense new minus old, will be

$$(49) \qquad \begin{aligned}
\delta x_1 &= [p_1 x_1 + P_2 x_2 + (P_3 - P_2')x_3 + (P_4 - P_3')x_4]\delta x, \\
\delta x_2 &= [- x_1 + p_1 x_2 + 2P_2 x_3 + 3(P_3 - P_2')x_4]\delta x, \\
\delta x_3 &= (- x_2 + p_1 x_3 + 3P_2 x_4)\delta x, \\
\delta x_4 &= (- x_3 + p_1 x_4)\delta x.
\end{aligned}$$

Referred to the new tetrahedron of reference, the equation of the complex osculating C_y at the point corresponding to $x + \delta x$ will be

$$(50) \qquad \bar{\omega}_{14} - 2\bar{P}_2 \bar{\omega}_{34} - \bar{\omega}_{23} = 0,$$

where

$$\bar{P}_2 = P_2 + P_2' \delta x, \qquad \bar{\omega}_{ik} = \bar{x}_i \bar{y}_k - \bar{x}_k \bar{y}_i,$$
$$\bar{x}_i = x_i + \delta x_i, \qquad \bar{y}_i = y_i + \delta y_i,$$

if $\bar{x}_i$ and $\bar{y}_i$ denote the coördinates of two points on a line of the complex referred to the new tetrahedron of reference. Making the calculations we find

$$\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} + [- \omega_{13} + 2p_1 \omega_{14} - P_2 \omega_{42} + (P_3 - P_2')\omega_{34}]\delta x, \\
\bar{\omega}_{34} &= \omega_{34} + (2p_1 \omega_{34} + \omega_{42})\delta x, \\
\bar{\omega}_{23} &= \omega_{23} + [2p_1 \omega_{23} - 3P_2 \omega_{42} - \omega_{13} - 3(P_3 - P_2')\omega_{34}]\delta x.
\end{aligned}$$

If we substitute in (50), we find as the *equation of the linear complex osculating C_y at a point infinitesimally close to P_y*,

$$(51)\qquad (\omega_{14}-2P_2\omega_{34}-\omega_{23})(1+2p_1\delta x)+4\theta_3\omega_{34}\delta x=0.$$

This coincides with the linear complex osculating C_y at P_y, if, and only if, $\theta_3=0$.

Therefore, *if the invariant θ_3 vanishes identically, the tangents of the curve C_y belong to a linear complex.* If it does not vanish identically, those values of x for which it does vanish correspond to points of the curve at which the osculating linear complex hyperosculates the curve.

This result, which has been known a long time, is here derived in a novel manner, which has above all the advantages of clearness. It may also be obtained by setting up the linear differential equation of the sixth order satisfied by the six line coördinates

$$\omega_{ik}=y_iy'_k-y_ky'_i,$$

of the tangent, and noting that this reduces to the fifth order if, and only if, $\theta_3=0$. This is the method of HALPHEN.

A former result may now be stated as follows: *The osculating and torsal cubics of all points of a curve coincide, if, and only if, the curve belongs to a linear complex.*

§ 5. *Geometrical definition of the fundamental tetrahedron of reference.*

We have seen that there exists for every point of the curve C_y a tetrahedron whose vertices P_y, P_z, P_ρ, P_σ are determined by the choice of the independent variables x. In order that we may be able to obtain a clear insight into the geometry of the curve, it is necessary that we may be able to define this tetrahedron by purely geometrical considerations. As a consequence of our preceding results we are now able to do this.

We have already noticed that P_z is a point on the tangent, and by a properly chosen transformation $\xi=\xi(x)$ it may be transformed into any other point of the tangent. When the independent variable has been definitely chosen to be x, we obtain, therefore, a point P_z on the tangent which is not, in general, distinguished by any geometrical property from any other point of the tangent. Its position may serve as a geometric image of the independent variable.

Consider the osculating conic

$$x_4=0,\qquad 40x_1x_3-32P_2x_3^2-15x_2^2=0.$$

The polar of any point $(x'_1, x'_2, x'_3, 0)$ of the osculating plane with respect to it, is the straight line

$$x_4=0,\qquad 20x'_3x_1-15x'_2x_2+(20x'_1-32P_2x'_3)x_3=0.$$

Therefore, the polar of P_z, whose coördinates are $(0, 1, 0, 0)$, is the line $x_2 = 0$, $x_4 = 0$. In other words:

The line $P_y P_\rho$ is the polar of P_z with respect to the osculating conic.

We shall speak of the curves C_z, C_ρ, C_σ as the derivative curves of C_y with respect to x, of the first, second and third kind respectively. The ruled surfaces which are obtained by joining the points of C_y to the corresponding points of C_z, C_ρ, C_σ shall be called derivative ruled surfaces of the first, second and third kind respectively. Then, the derivative ruled surface of the first kind is unique. It is simply the developable whose cuspidal edge is C_y. Let us consider the derivative ruled surface S of the second kind generated by $P_y P_\rho$. The curve C_y is, of course, an asymptotic curve upon it. This surface is characterized by the equations (20), where p_1 has been assumed equal to zero.

According to the general theory of ruled surfaces,* the asymptotic tangents to S at the points P_y and P_ρ are obtained by joining these points to $2z$ and

$$2\sigma - 4P_2 z + 2P_3 y$$

respectively. Therefore the asymptotic tangent to S at any point $(\alpha_1, 0, \alpha_3, 0)$ of $P_y P_\rho$ joins this point to

$$2P_3 \alpha_3 y + (2\alpha_1 - 4P_2 \alpha_3) z + 2\alpha_3 \sigma .$$

Hence, the equation of the plane tangent to S at $(\alpha_1, 0, \alpha_3, 0)$ is

$$-\alpha_3 x_2 + (\alpha_1 - 2P_2 \alpha_3) x_4 = 0 .$$

To the same point of $P_y P_\rho$ there corresponds a plane in the osculating linear complex. According to $(47a)$ this is the plane

$$\alpha_3 x_2 + (\alpha_1 - 2P_2 \alpha_3) x_4 = 0 .$$

Therefore, if at any point of the generator of the derived ruled surface of the second kind we construct the tangent plane as well as the plane which corresponds to it in the osculating linear complex, these planes form an involution. The double planes of this involution are the osculating plane $(x_4 = 0)$, and a plane $(x_2 = 0)$ which contains P_σ, the point of the derivative curve of the third kind which corresponds to P_y.

The point which corresponds to this latter plane, is

$$\beta = 2P_2 y + \rho . \tag{52}$$

According to $(47a)$ we have further, corresponding to the point P_z or $(0, 1, 0, 0)$, the plane $x_3 = 0$, which also contains P_σ. The line $P_y P_\sigma$ is now completely determined, as follows:

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* WILCZYNSKI, *Covariants*, etc. Transactions of the American Mathematical Society, vol. 3 (1902), p. 434.

The generator of the derived ruled surface of the third kind is the intersection of the following two planes: 1st, the plane corresponding to P_z in the osculating linear complex; 2d, that plane which is tangent to the derived ruled surface of the second kind at the same point which corresponds to it in the osculating linear complex.

It still remains to determine the position of P_ρ and P_σ on the lines $P_y P_\rho$ and $P_y P_\sigma$.

The osculating conic intersects $P_y P_\rho$ in P_y and in P_α where

$$\alpha = 4P_2 y + 5\rho. \tag{53}$$

The cross-ratio of the four points P_y, P_ρ, P_α, P_β, is

$$(\alpha, y, \beta, \rho) = \tfrac{5}{2}.$$

If, upon the generator of the derived ruled surface of the second kind, there be marked its intersections with the osculating conic, and the point P_β whose tangent plane coincides with the plane corresponding to it in the osculating linear complex, the point P_ρ is determined by the condition that the cross-ratio of these four points shall be equal to $\frac{5}{2}$.

If $P_2 = 0$ this definition of P_ρ breaks down. In that case, however, P_α and P_β coincide with P_ρ. Therefore, *if the derived ruled surface of the third kind is a developable, P_ρ is that point on the generator of the derived surface of the second kind where this generator intersects the osculating conic the second time. At this point the plane, tangent to the ruled surface, and the plane, corresponding to it in the osculating linear complex, coincide.*

If we use the notations of the theory of ruled surfaces,* we find from (20),

$$u_{12} = 4, \qquad u_{21} = 8P_3' - 4P_4 + 8P_2^2, \qquad u_{11} - u_{22} = 16P_2,$$
$$(u_{11} - u_{22})^2 + 4u_{12}u_{21} = -64(P_4 - 2P_3' - 6P_2^2).$$

But $u_{11} - u_{22} = 0$ is the condition that C_y and C_ρ shall be harmonically divided by the branches of the flecnode curves of the ruled surface, while $(u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0$ is the condition under which the two branches of the flecnode curve coincide.† Therefore, we obtain the following theorem.

If the derived ruled surface of the third kind is a developable, the intersections of the generator of the derived ruled surface of the second kind with the osculating conic give rise to two curves upon this surface harmonically conjugate with respect to the two branches of its flecnode curve.

If $\theta_4 = 0$, the second intersection of the generator of this surface with the osculating conic is a point of its flecnode curve. Moreover, the two branches of the flecnode curve must then coincide.

* WILCZYNSKI, *Covariants*, etc., loc. cit., p. 445. *Invariants*, etc., Transactions of the American Mathematical Society, vol. 2 (1901), p. 6.

† Loc. cit., p. 444.

It is to be noted that we have here a *geometrical interpretation for the invariant equation* $\theta_4 = 0$. We shall find two other, quite different interpretations for this condition later on. By the method of HALPHEN still another meaning can be attached to this equation, not however so purely geometrical.

We may, if we wish, make use of the torsal cubic in our further constructions. For, it is now defined entirely by geometrical considerations. If, in fact, we trace upon the developable, whose edge of regression is C_y an arbitrary curve $C_{\bar{z}}$, we now know how the corresponding ruled surfaces of the second and third kind may be constructed. They depend upon an arbitrary function of x, as does the curve $C_{\bar{z}}$. Among the surfaces of the third kind there exists a single one-parameter family of developables. Upon that generator of each of these developables which passes through P_y we mark the point where it intersects the cuspidal edge of the developable to which it belongs. The locus of these points is the torsal cubic.

We notice incidentally that *the reduction of equation* (1) *to the Laguerre-Forsyth canonical form is equivalent to the determination of one of the developables of the third kind.* Since this reduction is made by solving an equation of the RICCATI form we notice further the following theorem. *The four curves on the developable of C_y which correspond to any four of the developables of the third kind, intersect all of the tangents of C_y in point-rows of the same cross-ratio.*

Let us consider the developable surface of the torsal cubic, which is given by equations (44). We are going to find its intersection with the plane $P_y P_z P_\sigma$, or $x_3 = 0$. The intersection of the plane $u_1, \cdots, u_4$, which osculates the cubic at the point whose parameter is η, with the plane $x_3 = 0$ is the line

$$- 180x_1 + 270\eta x_2 + (540\theta_3 + 54P_2' - 324P_2\eta + 135\eta^3)x_4 = 0$$

of this plane. As η changes this line envelops a curve, the required intersection. Its equation will be found by eliminating η between the above equation and this other one

$$270x_2 + (-324P_2 + 405\eta^2)x_4 = 0$$

obtained from it by differentiation with respect to η. This elimination may be easily performed. The result is

$$(54) \quad F = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - (30\theta_3 + 3P_2')x_4\}^2 = 0.$$

This plane cubic together with the tangent $P_y P_z$ gives the complete intersection of the plane $P_y P_z P_\sigma$ with the developable of the torsal cubic. It has a cusp at P_z, and the equation of its cusp tangent is

$$5x_1 - 6x_4 = 0,$$

as one may find by the general theory of plane curves. It intersects $P_y P_\sigma$ in the point

$$\kappa = 24y + 20\sigma. \tag{55}$$

The tangent to the plane cubic at P_y is

$$10x_2 + 3x_4 = 0.$$

It intersects the cubic again in the point

$$(12 + 30\theta_3 + 3P_2')y - 6z + 20\sigma.$$

If this point be joined to P_z by a straight line, the latter will intersect $P_y P_\sigma$ in the point

$$\lambda = (12 + 30\theta_3 + 3P_2')y + 20\sigma. \tag{56}$$

The plane $P_y P_z P_\sigma$ is tangent to the torsal cubic. It intersects it once more in the point corresponding to $\eta = 0$, viz., $(60\theta_3 + 6P_2')y + 24P_2 z + 20\sigma$. A line joining this point to P_z intersects $P_y P_\sigma$ in

$$\mu = (60\theta_3 + 6P_2')y + 20\sigma. \tag{57}$$

Consider the four points P_κ, P_λ, P_μ and P_y. We have

$$\lambda = \frac{\mu + \kappa}{2}, \qquad (30\theta_3 + 3P_2' - 12)y = \frac{\mu - \kappa}{2};$$

so that P_λ is the harmonic conjugate of P_y with respect to P_κ and P_μ.

The osculating cubic differs from the torsal cubic only in having $2(P_3 - P_2')$ in place of $10\theta_3 + P_2'$. Consequently the plane cubic in which its developable intersects the plane $P_y P_z P_\sigma$ is

$$\bar{F} = 8(5x_2 - 6x_4)(5x_1 - 6x_4)^2 + 15x_4\{10x_1 - 6(P_3 - P_2')x_4\}^2 = 0. \tag{58}$$

If we denote by $P_{\bar\kappa}$, $P_{\bar\lambda}$, $P_{\bar\mu}$ the points constructed with respect to this curve in the same way as P_κ, and P_λ and P_μ were with respect to $F = 0$, we find

$$\bar\kappa = \kappa, \quad \bar\lambda = (12 + 6P_3 - 6P_2')y + 20\sigma, \quad \bar\mu = (12P_3 - 12P_2')y + 20\sigma, \tag{59}$$

the cusp and its tangent being common to the two curves, as well as the tangent at P_y. I refrain from formulating explicitly the various theorems which may be obtained from these equations.

In order to obtain a simple construction for P_σ, we shall consider finally the developable generated by the motion of the plane $P_y P_\rho P_\sigma$. The equation of this plane is $x_2 = 0$. As x changes into $x + \delta x$, y, z, ρ change into $y + y'\delta x$, $\rho + \rho'\delta x$, $\sigma + \sigma'\delta x$ respectively, where y', ρ', σ' are given by equations (5). The equation of the plane of these points, referred to the tetrahedron P_y, P_z, P_ρ, P_σ, is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ 1-p_1\delta x & \delta x & 0 & 0 \\ -(P_3-P_2')\delta x & -2P_2\delta x & 1-p_1\delta x & \delta x \\ -(P_4-P_3')\delta x & -3(P_3-P_2')\delta x & -3P_2\delta x & 1-p_1\delta x \end{vmatrix} = 0,$$

which becomes, when developed,

$$x_1\delta x - x_2(1-3p_1\delta x) - 2x_3P_2\delta x - 3x_4(P_3-P_2')\delta x = 0.$$

Therefore the equations of the generator of the developable, generated by the motion of the plane $P_yP_\rho P_\sigma$, are

$$(60) \qquad x_2 = 0, \qquad x_1 - 2P_2x_3 - 3(P_3-P_2')x_4 = 0.$$

It intersects the generator $P_yP_\rho(x_2 = x_4 = 0)$ of the derived ruled surface of the second kind in that point

$$(52) \qquad \beta = 2P_2y + \rho$$

whose tangent plane coincides with the plane corresponding to it in the osculating linear complex. Its intersection with P_yP_σ, the generator of the derived ruled surface of the third kind, is

$$(61) \qquad \gamma = 3(P_3-P_2')y + \sigma.$$

The generator of the developable joins P_β to P_γ. We wish to determine its edge of regression. If

$$\delta = l\beta + m\gamma$$

is the point where $P_\beta P_\gamma$ meets the edge of regression, we must have

$$\delta' = r\beta + s\gamma, \qquad \text{or} \qquad l\beta' + m\gamma' = \bar{r}\beta + \bar{s}\gamma.$$

We proceed to determine the ratio of l to m. We find

$$\beta' = (3P_2' - 2p_1P_2 - P_3)y - p_1\rho + \sigma,$$

$$\gamma' = -[P_4 - 4P_3' + 3P_2'' + 3p_1(P_3-P_2')]y - 3P_2\rho - p_1\sigma.$$

We may eliminate ρ and σ by (52) and (61). This gives

$$\beta' = -4\theta_3y - p_1\beta + \gamma,$$

$$\gamma' = -(P_4 - 4P_3' + 3P_2'' - 6P_2^2)y - 3P_2\beta - p_1\gamma.$$

We may therefore put

$$l = P_4 - 4P_3' + 3P_2'' - 6P_2^2, \qquad m = -4\theta_3;$$

so that

$$(62) \qquad \delta = (P_4 - 4P_3' + 3P_2'' - 6P_2^2)\beta - 4\theta_3\gamma$$

gives the edge of regression. This gives the following theorem:

The developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, has its edge of regression upon the derived ruled surface of the second kind, if, and only if, the curve C_y *belongs to a linear complex.*

We may write in place of (51)

$$\gamma = 60(P_3 - P_2')y + 20\sigma.$$

We have from (59)

$$\bar{\mu} = 12(P_3 - P_2')y + 20\sigma.$$

Therefore, the cross-ratio of the four points P_γ, P_y, $P_{\bar{\mu}}$ and P_σ is

$$(\gamma, y, \bar{\mu}, \sigma) = 5.$$

We have found finally a geometrical definition for P_σ, which we may recapitulate as follows. *The plane of the tangent and the generator of the third derived ruled surface intersects the osculating cubic in* P_y *counted twice and one other point. If the latter point be joined to* P_z *by a straight line we obtain a certain point* $P_{\bar{\mu}}$ *as the intersection of this line with* P_yP_σ. *The generator of the developable, generated by the plane of the generators of the derived ruled surfaces of the second and third kind, intersects* P_yP_σ *in another point* P_γ. P_σ *may now be found as that point of* P_yP_σ *which makes the cross-ratio*

$$(P_\gamma, P_y, P_{\bar{\mu}}, P_\sigma) = 5.$$

We have shown how to construct the fundamental tetrahedron when P_z is given. If P_ρ is given, P_z can be found at once as the pole of P_yP_ρ with respect to the osculating conic. If P_σ is given we may find first its polar plane with respect to the osculating linear complex which is

$$-x_1 + 2P_2x_3 = 0,$$

and therefore passes through P_z, but not through P_y. P_z can therefore be found at once at the intersection of this plane with the tangent to C_y at P_y.

We see therefore, that *any one of the three points* P_z, P_ρ, P_σ *determines uniquely the others.*

§ 6. *Some further properties of the derived ruled surfaces of the second and third kind.*

Let us suppose $p_1 = P_2 = 0$, so that the derived ruled surface of the third kind is a developable, and let us consider the derived ruled surface of the second kind which corresponds to it. We proceed to deduce the equation of its osculating linear complex.

Let Y and R denote the developments of y and ρ in the vicinity of the ordi-

nary point $x = a$, and replace again $x - a$ by x in the developments. Then we shall find

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma, \qquad R = \rho_1 y + \rho_2 z + \rho_3 \rho + \rho_4 \sigma,$$

where

$$(63) \qquad \begin{aligned} \rho_1 &= -P_3 x - \tfrac{1}{2} P_4 x^2 - \tfrac{1}{6} P_4' x^3 + \tfrac{1}{24}(4P_3^2 - P_4'')x^4 + \cdots, \\ \rho_2 &= -2P_3 x^2 - \tfrac{1}{6}(4P_3' + P_4)x^3 - \tfrac{1}{12}(2P_3'' + P_4')x^4 + \cdots, \\ \rho_3 &= 1 - \tfrac{2}{3} P_3 x^3 - \tfrac{1}{24}(8P_3' + P_4)x^4 + \cdots, \\ \rho_4 &= x - \tfrac{1}{6} P_3 x^4 + \cdots, \end{aligned}$$

while $y_1, \cdots, y_4$ have been computed before. Denote by

$$\omega_{ik} = y_i \rho_k - y_k \rho_i$$

the Plückerian coördinates of the line joining the two points. We find

$$\begin{aligned} \omega_{12} &= -P_3 x^2 - \tfrac{1}{3}(2P_3' - P_4)x^3 - \tfrac{1}{12}(2P_3'' - P_4')x^4 + \cdots, \\ \omega_{13} &= 1 - \tfrac{1}{3} P_3 x^3 - \tfrac{1}{6}(2P_3' - P_4)x^4 + \cdots, \\ \omega_{14} &= x - \tfrac{1}{6} P_3 x^4 + \cdots, \\ \omega_{23} &= x + \tfrac{1}{6} P_3 x^4 + \cdots, \\ \omega_{24} &= x^2 + \cdots, \qquad \omega_{34} = \tfrac{1}{3} x^3 + \cdots. \end{aligned}$$

Let

$$a\omega_{12} + b\omega_{12} + c\omega_{14} + d\omega_{23} + e\omega_{24} + f\omega_{34} = 0$$

be the equation of the osculating linear complex of the surface in question. Then, the coefficients of all powers of x up to and including x^4 must be zero, if we substitute the above developments of ω_{ik} into the left member. This gives us the following equations:

$$\begin{aligned} b = 0, \qquad c + d = 0, \qquad -P_3 a + e &= 0, \\ -\tfrac{1}{3}(2P_3' - P_4)a - \tfrac{1}{3}P_3 b + \tfrac{1}{3} f &= 0, \\ -\tfrac{1}{12}(2P_3'' - P_4')a - \tfrac{1}{6}(2P_3' - P_4)b - \tfrac{1}{6}P_3 c + \tfrac{1}{6}P_3 d &= 0, \end{aligned}$$

whence the ratios of the coefficients may be easily deduced.

We find thus the equation of the linear complex osculating the derived ruled surface of the second kind which corresponds to a developable of the third kind; it is

$$(64) \quad -4P_3\omega_{12} - (P_4' - 2P_3'')(\omega_{14} - \omega_{23}) - 4P_3^2\omega_{24} + 4(P_4 - 2P_3')P_3\omega_{34} = 0.$$

It coincides with the osculating linear complex of C_y if and only if $P_3 = 0$, i. e., if C_y belongs to a linear complex. This result is also obvious for geometrical reasons.

The coördinates v_k of the plane, which corresponds to a point x_1, x_2, x_3, x_4 in the linear complex (64), are given by

$$
\begin{aligned}
&\omega v_1 = 4P_3x_2 + (P_4' - 2P_3'')x_4,\\
&\omega v_2 = -4P_3x_1 - (P_4' - 2P_3'')x_3 + 4P_3^2x_4,\\
&\omega v_3 = (P_4' - 2P_3'')x_2 - 4P_3(P_4 - 2P_3')x_4,\\
&\omega v_4 = -(P_4' - 2P_3'')x_1 - 4P_3^2x_2 + 4P_3(P_4 - 2P_3')x_3,
\end{aligned}
\tag{65}
$$

where ω is a proportionality factor.

Let us consider at the same time the osculating linear complex of C_y. The lines common to the two complexes form a congruence whose directrices we propose to find. This we can do quite easily by writing down the equations which express that, for a point on one of the directrices, the two planes corresponding to it in the two complexes must coincide. The right members of (65) must, for such a point be equal to

$$-\omega x_4, \qquad +\omega x_3, \qquad -\omega x_2, \qquad +\omega x_1$$

respectively, where ω is a proportionality factor.

The four equations obtained in this way can be satisfied only if their determinant vanishes, which gives

$$2P_3'' - P_4' - \omega = \pm 4P_3\sqrt{2P_3' - P_4}, \tag{66}$$

whence the following equations for the two directrices

$$
\begin{aligned}
\pm\sqrt{2P_3' - P_4}\,x_1 - P_3x_2 - (2P_3' - P_4)x_3 &= 0,\\
-x_2 \pm \sqrt{2P_3' - P_4}\,x_4 &= 0,\\
-x_1 \pm \sqrt{2P_3' - P_4}\,x_3 + P_3x_4 &= 0,
\end{aligned}
\tag{67}
$$

of which three equations only two are independent, and where we have assumed $P_3 \neq 0$. In fact, if P_3 were zero the two complexes would coincide and the congruence would be indeterminate.

Since we have assumed $P_2 = 0$, the quantity under the square root is $-\theta_4$. We find a second interpretation for the condition $\theta_4 = 0$. *If $\theta_4 = 0$, the congruence has coincident directrices.* We may combine this with our former result to the following theorem.

Choose as derived ruled surface of the third kind one of the developables of the single family which exists. Consider the osculating linear complex of the corresponding ruled surface S of the second kind. Let the directrices of the congruence, which this complex has in common with the osculating linear complex of the fundamental curve C_y, coincide. Then the two branches of the

flecnode curve of S coincide and the generator of S which passes through P_y will intersect the osculating conic of C_y in P_y, and a second point whose locus is the flecnode curve of the surface S.

I refrain from formulating the converse. The above conditions are fulfilled if and only if $\theta_4 = 0$.

Let us consider one of the directrices (67), for example that one which corresponds to the plus sign of the square root. Then we see that

$$\tag{68} \begin{aligned} \alpha &= P_3\sqrt{2P_3' - P_4}\,y + P_3\rho, \\ \beta &= \sqrt{2P_3' - P_4}\,(P_3 y + \sqrt{2P_3' - P_4}\,z + \sigma) \end{aligned}$$

are two points on the directrix. We have multiplied each expression by a factor so as to have α and β of the same weight. If now we change the independent variable, but in such a way as not to disturb the condition $P_2 = 0$, we shall get in (64) a single infinity of complexes, and in (67) two families of lines, the directrices of the single infinity of congruences which thus result. We are going to study to some extent the two ruled surfaces thus generated.

Put

$$\tag{69} k = \sqrt{2P_3'' - P_4}.$$

Making the transformations which preserve $P_2 = 0$, for which we must have

$$\mu = \eta' - \tfrac{1}{2}\eta^2 = 0,$$

we find that α and β are transformed into $\bar\alpha$ and $\bar\beta$, where

$$\begin{aligned} (\xi')^5\bar\alpha &= \alpha + 2\eta P_3 z + \tfrac{3}{2}\eta^2 P_3 y, \\ (\xi')^5\bar\beta &= \beta + \tfrac{3}{2}\frac{k}{P_3}\eta\alpha + k(\tfrac{3}{2}\eta^2 z + \tfrac{3}{4}\eta^3 y). \end{aligned}$$

The point $m\bar\alpha + n\bar\beta$ will be an arbitrary point on the line joining $P_{\bar\alpha}P_{\bar\beta}$. We find therefore, the equations of our surface referred to two parameters η and m/n,

$$\tag{70} \begin{aligned} x_1 &= \tfrac{3}{2}mP_3\eta^2 + (m+n)P_3k + n(\tfrac{3}{2}\eta k^2 + \tfrac{3}{4}\eta^3 k), \\ x_2 &= 2m\eta P_3 + n(k^2 + \tfrac{3}{2}\eta^2 k), \\ x_3 &= mP_3 + \tfrac{3}{2}kn\eta, \\ x_4 &= nk. \end{aligned}$$

The nature of this surface may be easily determined. We have, returning to the two curves C_α and C_β upon it, for C_α

$$\alpha_1 = k + \tfrac{3}{2}\eta^2, \qquad \alpha_2 = 2\eta, \qquad \alpha_3 = 1, \qquad \alpha_4 = 0,$$

if we put $n = 0$, $m = 1/P_3$ in (70). If we put $m = 0$, $n = 1/k$ we find for C_β

$$\beta_1 = P_3 + \tfrac{3}{2}k\eta + \tfrac{3}{4}\eta^3, \qquad \beta_2 = k + \tfrac{3}{2}\eta^2, \qquad \beta_3 = \tfrac{3}{2}\eta, \qquad \beta_4 = 1.$$

But (α_k, β_k) are simultaneous solutions of the equation

$$\frac{d\beta}{d\eta} = \tfrac{3}{2}\alpha,$$

which proves that *the ruled surface which we are considering is a developable whose edge of regression is the twisted cubic* C_β.

The curve C_α is a conic, the intersection of the developable of the cubic with the osculating plane of C_y. Its equations are

$$x_4 = 0, \qquad -3x_2^2 + 8x_1x_3 - 8kx_3^2 = 0.$$

We notice that for $\theta_4 = 0$ *it coincides with the osculating conic*, a further interpretation of this condition. *In general, the two conics have a contact of third order at* P_y.

If, in these equations, we change k into $-k$ we obtain the developable, cubic and conic associated with the second directrix of our congruence. A considerable number of other configurations are suggested by the combinations of these various curves and surfaces. I will refrain, however, from any further study in this direction.

The curve C_y is an asymptotic curve upon every derived ruled surface of the second kind. Moreover, the most general derived ruled surface of the second kind depends upon one arbitrary function, as does also the most general ruled surface containing C_y as an asymptotic curve. It is easy to see that the derived ruled surface of the second kind may be made to coincide with any ruled surface upon which C_y is an asymptotic curve if the independent variable be properly chosen.

Upon the derived ruled surface of the third kind, C_y can never be an asymptotic curve. It may, however, be one branch of the flecnode curve. In fact, if we form the quantities u_{ik} of the theory of ruled surfaces for system (27), we find

$$u_{12} = -\frac{P_2'}{P_2^2}.$$

But $u_{12} = 0$ is the condition that C_y may be a branch of the flecnode curve on the surface generated by $P_y P_\sigma$. Suppose that the variable has been so chosen as to make $P_2' = 0$. The most general transformation which is possible, leaving this relation invariant, must, according to (11) satisfy the condition

$$-2\eta P_2 - \tfrac{5}{6}\mu' + \tfrac{5}{3}\eta\mu = 0,$$

or

$$-5\eta'' - 12\eta P_2 + 15\eta\eta' - 5\eta^3 = 0, \tag{71}$$

a differential equation of the second order for η. Moreover two different solutions of this equation always give rise to two distinct ruled surfaces. For, let

η_1 and η_2 be two such solutions, and let σ_1, σ_2 be the corresponding values of σ. Then, according to (9),

$$\sigma_\kappa = \frac{1}{(\xi'_\kappa)^3}\left[\sigma + \tfrac{3}{2}\eta_\kappa \rho + (\mu_\kappa + \tfrac{3}{2}\eta_\kappa^2)z + \tfrac{1}{4}(\mu'_\kappa + 4\eta_\kappa\mu_\kappa + 3\eta_\kappa^3)y\right] \quad (\kappa = 1, 2),$$

But if the same ruled surface corresponds to η_1 and η_2, the three points y, σ_1 and σ_2 must be collinear. We must therefore be able to reduce

$$(\xi'_1)^3\sigma_1 - (\xi'_2)^3\sigma_2 = \tfrac{3}{2}(\eta_1 - \eta_2)\rho + \cdots$$

to a multiple of y. But this is clearly only possible if $\eta_2 = \eta_1$.

We see, therefore, that *there are ∞^2 derived ruled surfaces of the third kind upon which C_y is one branch of the flecnode curve.*

If P'_2 is not zero, our problem leads to the differential equation

$$(72) \qquad \frac{d}{dx}(\eta' - \tfrac{1}{2}\eta^2) = \tfrac{6}{5}P'_2 - \tfrac{12}{5}P_2\eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2),$$

which is of the second order and third degree.

I have shown in a former paper that the most general ruled surface which has C_y as one branch of its flecnode curve contains an arbitrary function in its general expression.* I have also shown that together with any such surface, its flecnode surface and each member of a single infinity of surfaces determined by these two, also contains C_y as one branch of its flecnode curve. One might imagine that there could be based upon these theorems a transformation theory of equation (72). This is not the case however. For, if one of the surfaces containing C_y as a branch of its flecnode curve is a derived ruled surface of the third kind, its flecnode surface is not, nor is any member of the family of ruled surfaces just mentioned.

Corresponding to the ∞^2 solutions of (72) or of

$$\mu' = \tfrac{6}{5}P'_2 - \tfrac{12}{5}P_2\eta + 2\eta\mu,$$

we find ∞^2 positions for $P_{\bar\sigma}$, viz.:

$$4(\xi')^3\bar\sigma = (\tfrac{6}{5}P'_2 - \tfrac{12}{5}P_2\eta + 6\eta\mu + 3\eta^3)y + (4\mu + 6\eta^2)z + 6\eta\rho + 4\sigma.$$

The locus of these points is a cubic surface

$$(73) \qquad 27P'_2x_4^3 - 36P_2x_3x_4^2 + 90x_2x_3x_4 - 90x_1x_4^2 - 40x_3^3 = 0,$$

which contains P_yP_z, the tangent of C_y as a double line. It is therefore a ruled surface. It is in fact a CAYLEY's cubic scroll.

If one derived surface of the third kind is known upon which C_y is a branch

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* *General theory of curves on ruled surfaces,* Transactions of the American Mathematical Society, vol. 6 (1905), p. 78.

of the flecnode curve, two others may be found by merely solving a quadratic equation.

In fact, suppose that a solution η of (72) be known. We may make a transformation of the independent variables, $\xi = \xi(x)$ such that

$$\frac{\xi''}{\xi'} = \eta.$$

In the resulting equation $\overline{P}_2' = 0$. If we again denote the independent variable by x, (72) becomes

$$\eta'' - \eta\eta' = -\tfrac{12}{5}P_2\eta + 2\eta(\eta' - \tfrac{1}{2}\eta^2),$$

where P_2 is a constant, since $P_2' = 0$. But we may satisfy this equation by putting $\eta =$ const., which gives the equation

$$\eta^3 + \tfrac{12}{5}P_2\eta = 0,$$

whence

$$\eta = 0, \qquad \pm\sqrt{-\tfrac{12}{5}P_2}.$$

The root $\eta = 0$ gives the original solution. The other two are new.

§ 7. *The principal tangent plane of two space curves. The covariants. Transition to Halphen's investigations.*

HALPHEN has introduced a very important notion, which we shall now proceed to explain.

Let there be given two space curves having at a point P a contact of the nth order. If these curves be projected from any center Q upon a plane, the projections will also have, in general, a contact of the nth order at the point corresponding to P. HALPHEN shows that there exists a plane passing through the common tangent of the two curves such that if the center of projection be taken anywhere within it, the contact of the projections will be of an order higher than n. This plane he calls the *principal tangent plane* of the two curves.*

We shall follow HALPHEN in determining the principal tangent plane at P_y of the curve C_y and its osculating cubic. This will lead us to an especially simple form for the development of the equations of the curve, which is also due to HALPHEN and on the basis of which he draws his further conclusions. It will also enable us to substitute for our system of covariants C_2, C_3, C_4 another system whose geometrical significance will be apparent, and in terms of which C_2, C_3, C_4 may be expressed.

* HALPHEN, *Sur les invariants différentiels des courbes gauches*, Journal de l'Ecole Polytechnique, vol. 47 (1880), p. 25.

Assuming $P_2 = 0$, the equations of the osculating cubic, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, are

$$3x_2x_4 - 2x_3^2 = 0, \qquad 2x_3(5x_1 - 3P_3x_4) - 5x_2^2 = 0.$$

Let us put

$$(74) \qquad \bar{x}_1 = x_2, \qquad \bar{x}_2 = \tfrac{2}{5}x_3, \qquad \bar{x}_3 = \tfrac{6}{25}x_4, \qquad \bar{x}_4 = 5x_1 - 3P_3x_4,$$

and

$$(75) \qquad x = \frac{\bar{x}_1}{\bar{x}_4}, \qquad y = \frac{\bar{x}_2}{\bar{x}_4}, \qquad z = \frac{\bar{x}_3}{\bar{x}_4},$$

Then the equations of the cubic reduce to

$$(76) \qquad y = x^2, \qquad z = x^3.$$

The relation of the new tetrahedron of reference to the cubic is quite simple. The plane $\bar{x}_3 = 0$ is the osculating plane at P; $\bar{x}_2 = 0$ is some other plane through the tangent; this plane intersects the cubic in another point Q; the plane tangent to the cubic at Q and passing through P is $\bar{x}_1 = 0$; the osculating plane at Q is $\bar{x}_4 = 0$.

Since $\bar{x}_2 = 0$ may be chosen in an infinity of ways we see that the reduction of the equations of a space cubic to the form (76) may be accomplished in an infinity of ways.

For the curve C_y we have

$$(77) \qquad \begin{aligned}
y_1 &= 1 - \frac{P_3}{3!}x^3 - \frac{P_4}{4!}x^4 - \frac{P_4'}{5!}x^5 - \frac{P_4'' - 4P_3^2}{6!}x^6 \\
&\qquad - \frac{P_4^{(3)} - 5P_3P_4 - 12P_3P_3'}{7!}x^7 + \cdots, \\
y_2 &= x - \frac{P_3}{3!}x^4 - \frac{P_4 + 4P_3'}{5!}x^5 - \frac{2P_4' + 4P_3''}{6!}x^6 \\
&\qquad + \frac{16P_3^2 - 3P_4'' - 4P_3^{(3)}}{7!}x^7 + \cdots, \\
y_3 &= \tfrac{1}{2}x^2 - \frac{4P_3}{5!}x^5 - \frac{1}{6!}(P_4 + 8P_3')x^6 - \frac{1}{7!}(3P_4' + 12P_3'')x^7 + \cdots, \\
y_4 &= \tfrac{1}{6}x^3 - \frac{4}{6!}P_3x^6 - \frac{1}{7!}(P_4 + 12P_3')x^7 \cdots.
\end{aligned}$$

If we refer it to the same tetrahedron to which we have just referred the osculating cubic, and if we put

$$\xi = \frac{\bar{y}_1}{\bar{y}_4}, \qquad \eta = \frac{\bar{y}_2}{\bar{y}_4}, \qquad \zeta = \frac{\bar{y}_3}{\bar{y}_4},$$

we shall find

$$(78)\quad \begin{aligned} \eta &= \xi^2 + \frac{125}{36}(8P_3' - 5P_4)\xi^6 + \frac{625}{7.36}(8P_3'' - 5P_4')\xi^7 + \cdots, \\ \zeta &= \xi^3 - \frac{25}{3}P_3\xi^6 + \frac{125}{7.12}(36P_3' - 25P_4)\xi^7 + \cdots. \end{aligned}$$

If we put

$$(79)\qquad \xi = \epsilon X, \qquad \eta = \epsilon^2 Y, \qquad \zeta = \epsilon^3 Z, \qquad \epsilon^3 = -\frac{3}{25P_3},$$

assuming, therefore, that P_3 is not zero, this becomes

$$(80)\quad \begin{aligned} Y &= X^2 + \lambda_6 X^6 + \lambda_7 X^7 + \cdots, \\ Z &= X^3 + \mu_6 X^6 + \mu_7 X^7 + \cdots, \end{aligned}$$

where

$$(81)\quad \begin{aligned} \lambda_6 &= \tfrac{125}{36}(8P_3' - 5P_4)\epsilon^4, \qquad & \lambda_7 &= \frac{625}{7.36}(8P_3'' - 5P_4')\epsilon^5, \\ \mu_6 &= 1, & \mu_7 &= \frac{125}{7.12}(36P_3' - 25P_4)\epsilon^4. \end{aligned}$$

But we can make a further change of coördinates without disturbing the form (80) of the development. For the plane $\bar{x}_2 = 0$ may be taken to be any plane through the tangent. It may be taken in such a way as to make λ_6 vanish, as we shall presently see. The plane thus obtained will obviously be the principal tangent plane of the curve C_y and its osculating cubic.

Instead of working out the transformation geometrically we shall put with HALPHEN

$$(82a)\qquad \bar{X} = \frac{\xi_1}{\omega_1}, \qquad \bar{Y} = \frac{\eta_1}{\omega_1}, \qquad \bar{Z} = \frac{Z_1}{\omega_1},$$

where

$$(82b)\quad \begin{aligned} &\omega_1 = 1 + 3pX + 3p^2Y + p^3Z, \qquad \eta_1 = Y + pZ, \\ &\xi_1 = X + 2pY + p^2Z, \end{aligned}$$

the quantity p being arbitrary. We shall find with him

$$(83)\quad \begin{aligned} \bar{Y} &= \bar{X}^2 + \Lambda_6\bar{X}^6 + \Lambda_7\bar{X}^7 + \cdots, \\ \bar{Z} &= \bar{X}^3 + M_6\bar{X}^6 + M_7\bar{X}^7 + \cdots, \end{aligned}$$

as the result of applying this transformation to any system of equations of the form (80). The coefficients of (83) are

$$(84)\quad \begin{aligned} M_6 &= \mu_6, \qquad & M_7 &= \mu_7 + 3p\mu_6, \\ \Lambda_6 &= \lambda_6 + p\mu_6, & \Lambda_7 &= \lambda_7 + p\mu_7 + p(p\mu_7 - \lambda_6). \end{aligned}$$

For

$$p = -\frac{\lambda_6}{\mu_6},$$

this reduces to the canonical form required, in which we may also assume $\mu_6 = 1$ if the curve does not belong to a linear complex.

If we combine the various transformations which we have made successively we get the following result: *If we introduce non-homogeneous coördinates by putting*

$$X = \frac{\bar{y}_1}{\bar{y}_4}, \qquad Y = \frac{\bar{y}_2}{\bar{y}_4}, \qquad Z = \frac{\bar{y}_3}{\bar{y}_4}, \tag{85a}$$

where

$$\begin{aligned} \bar{y}_1 &= 25\epsilon^2 y_2 + 20p\epsilon y_3 + 6p^2 y_4, \\ \bar{y}_2 &= 10\epsilon y_3 + 6py_4, \\ \bar{y}_3 &= 6y_4, \\ \bar{y}_4 &= 125\epsilon^3 y_1 + 75p\epsilon^2 y_2 + 30p^2\epsilon y_3 + (6p^3 - 75P_3\epsilon^3)y_4, \end{aligned} \tag{85b}$$

and where

$$\epsilon^3 = -\frac{3}{25P_3}, \qquad p = -\tfrac{125}{36}(8P_3' - 5P_4)\epsilon^4, \tag{85c}$$

the development of the equations of the curve C_y may be written in the canonical form

$$\begin{aligned} Y &= X^2 + \Lambda_7 X^7 + \cdots, \\ Z &= X^3 + X^6 + M_7 X^7 + \cdots, \end{aligned} \tag{86}$$

where $\Lambda_7, M_7, \cdots$ are given by (84) *together with* (81). *This transformation is valid if C_y does not belong to a linear complex.*

The coefficients of (86) are absolute invariants. We find, in fact,

$$\Lambda_7 = \frac{625\epsilon^5}{7.12\cdot 36\theta_3}(25\theta_4^2 + 20\theta_8 - 4\theta_{3.1}), \qquad M_7 = \frac{625\epsilon^4}{42}\theta_4, \tag{87}$$

where

$$\theta_8 = 4\theta_4\theta_3' - 3\theta_3\theta_4'. \tag{88}$$

From (86) and (76) it is clear that the plane $Y = 0$ or $\bar{y}_2 = 0$ is the principal plane of the curve C_y and its osculating cubic. In the original system of coördinates its equation will, therefore, be

$$4P_3x_3 + (8P_3' - 5P_4)x_4 = 0. \tag{89}$$

If C_y belongs to a linear complex, it coincides with the osculating plane.

The point which corresponds to (89) in the osculating linear complex is

$$(5P_4 - 8P_3')y + 4P_3z, \tag{90}$$

or in invariant form

$$(\theta_3' + \tfrac{5}{2}\theta_4)y + 2\theta_3 z, \tag{91}$$

an expression whose covariance may be verified directly. We shall call this point the *principal point of the tangent.* We thus obtain a curve on the developable of C_y which may be called *its principal curve.*

If we make a transformation such that

$$3\eta = \theta_3' + \tfrac{5}{2}\theta_4, \tag{92}$$

the point $P_{\bar{z}}$ describes the principal curve on the developable of C_y. The points $P_{\bar{\rho}}$ and $P_{\bar{\sigma}}$ also describe perfectly definite curves, whose expressions may be obtained from (9) by substituting for η the expression (92). *Any covariant may be expressed in terms of y and the three which we have just determined, which may, therefore, serve to replace the covariants C_2, C_3 and C_4.*

If C_y belongs to a linear complex these four curves all coincide, so that a different set of fundamental covariants must then be selected.

In this exceptional case our fundamental tetrahedron $P_y P_z P_\rho P_\sigma$ gives rise to a most remarkable configuration. If we put again $P_2 = 0$, we have in this case also $P_3 = 0$. $P_y P_\sigma$ generates one of the developables of the third kind of which C_σ is the cuspidal edge, while $P_y P_z$ of course generates the developable of which C_y is the cuspidal edge. The surface generated by $P_y P_\rho$ is not developable. Its equations become

$$\rho'' + P_4 y = 0, \qquad y'' - \rho = 0.$$

It belongs to the same linear complex as C_y, and C_y and C_ρ are the two branches of its complex curve which is at the same time an asymptotic curve. $P_z P_\rho$ generates a developable, since $z' = \rho$, of which C_z is the cuspidal edge. $P_\rho P_\sigma$ generates a developable, since $\rho' = \sigma$, of which C_ρ is the cuspidal edge. Finally $P_z P_\sigma$ generates a ruled surface whose equations are

$$\sigma'' - \frac{P_4'}{P_4}\sigma' + P_4 z = 0, \qquad z'' - \sigma = 0,$$

upon which C_z and C_σ are asymptotic lines.

We find, therefore, the following theorem.

If the curve belongs to a linear complex, we may, in an infinity of ways, choose the fundamental tetrahedron so that four of its edges give rise to developables whose cuspidal edges are described by the four vertices. The other two edges of the tetrahedron will then give rise to ruled surfaces upon each of which the vertices of the tetrahedron trace a pair of asymptotic curves. The latter coincide with the two branches of the complex curve for the derived surface of the second kind.

It does not seem convenient to develop the geometrical theory of the covariants for this case at the present moment, as it will appear naturally in connec-

tion with some further investigations which I have not yet completed, and which I hope to present on some future occasion.

Those results which HALPHEN has deduced for the general case, and which have not been mentioned in this paper might now be directly transcribed from his memoir. For we have connected our own theory with his by obtaining the canonical form (86) for the development of the equations of the curve. In the case $\theta_3 = 0$, a different canonical form must be used, which has also been indicated by HALPHEN, but not extensively employed. The discussion of this we shall leave for a later occasion together with the investigations mentioned above.

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